KIT Department of Informatics
Institute for Anthropomatics and Robotics (IAR)
High Performance Humanoid Technologies (H^2T)

Robotics I, WS 2024/2025

1. Exercise Sheet

Prof. Dr.-Ing. Tamim Asfour
Adenauerring 12, Geb. 50.19
Web: https://www.humanoids.kit.edu/

Date: October 30, 2024

Exercise 1 (Euler Angles, RPY Angles, Quaternions)

1. Let R_1 be a general 3×3 rotation matrix,

$$R_1 = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix},$$

   i.) Calculate the Euler angles $\mathbf{z}\mathbf{x}'\mathbf{z}''$ corresponding to R_1.

   ii.) Calculate the RPY angles (**xyz**-Konvention) corresponding to R_1.

2. Let R_2 be a rotation matrix, given by

$$R_2 = \begin{pmatrix} 0.36 & 0.48 & -0.8 \\ -0.8 & 0.6 & 0.0 \\ 0.48 & 0.64 & 0.6 \end{pmatrix}.$$

   Calculate the quaternion $\mathbf{q}$ that describes the rotation given by R_2.

Exercise 2 (Homogeneous Matrices)

Let $T \in \mathrm{SE}(3)$ be a homogeneous transformation matrix and $\mathbf{v} = (1, 2, 3)^\top$ be a vector, with

$$T = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1. Which transformation is described by T?
2. Apply the transformation described by T to $\mathbf{v}$.
3. Determine T^{-1}, being the inverse transformation matrix of T.

Exercise 3 (Concatenation of Coordinate Transformations)

We consider a service robot with a holonomic platform. The robot's x axis points in the direction of motion, and the z axis points upwards. The y axis is defined so that the coordinate system is right-handed. Let the initial pose of the robot in the basis coordinate system (BCS) be defined as

$$T_{\text{init}} = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following commands are consecutively sent to the service robot and executed:

1. Rotate around the z axis by $90°$.
2. Drive straight for 4 unit lengths.
3. Drive straight for 2 unit lengths, to the right for 3 unit lengths and finally rotate around the z axis by $-45°$.

Calculate the transformation matrices corresponding to the individual commands, and the final pose of the robot in BCS.

Exercise 4 (Distance Between Poses)

The current pose T_{TCP} and the target pose T_{Goal} of an endeffector are given as

$$T_{\text{TCP}} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } T_{\text{Goal}} = \begin{pmatrix} 0 & 0 & -1 & 7 \\ 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate the translational and rotational difference between T_{TCP} and T_{Goal}.

Exercise 5 (Quaternions)

Given are a point $\mathbf{p} = (5, 1, 7)^\top$, a vector $\mathbf{a} = (0, 0, 1)^\top$ and an angle $\phi = 90°$.

1. Represent $\mathbf{p}$ as a quaternion $\mathbf{v}$.
2. Determine the quaternion $\mathbf{q}$ that describes the rotation by an angle of ϕ around the axis $\mathbf{a}$. Also determine the $\mathbf{q}^*$, i.e., the conjugated quaternion of $\mathbf{q}$.
3. Transform the point $\mathbf{p}$ by $\mathbf{q}$ and determine the resulting point $\mathbf{p}'$.
4. Let $\mathbf{q}_1 = \left(\cos\frac{\pi}{2}, \mathbf{a}_1 \sin\frac{\pi}{2}\right)$ and $\mathbf{q}_2 = \left(\cos\frac{\pi}{2}, \mathbf{a}_2 \sin\frac{\pi}{2}\right)$ be quaternions with $\mathbf{a}_1 = (1, 0, 0)^\top$ and $\mathbf{a}_2 = (0, 1, 0)^\top$.

   Give the direct formulation of the SLERP interpolation between $\mathbf{q}_1$ and $\mathbf{q}_2$, depending on the parameter $t \in [0, 1]$. Provide the interpolation result for $t = \frac{1}{2}$.

Exercise 6 (Quaternions)

Show that the space $\mathbb{S}^3$ of unit quaternions is a subgroup of the quaternions $\mathbb{H}$.

Remark: G is a group $(G, \cdot)$ if and only if:

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$
2. Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. Identity Element: $\exists\, e \in G : \forall a \in G : e \cdot a = a \cdot e = a$
4. Inverse Element: $\forall a \in G : \exists\, a^{-1} : a \cdot a^{-1} = e$

Exercise 7 (Rotations and Machine Learning)

Particularly in Robotics, rotations can be input or output of learned models (e.g., when estimating object poses from camera images). In such cases, a suitable representation of rotations needs to be chosen.

(The following questions are phrased rather open, and are intended as food for thought)

1. Compare the representation of rotations as Euler angles, quaternions and rotation matrices with respect to how suitable they are as the output of a machine learning approach (e.g., neural networks). Which properties of the representations might be advantegeous, and which ones disadvantegeous?

2. A neural network, which has been trained to output rotation matrices, yields the matrix A:

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix} .$$

   From A, a valid rotation matrix R is to be determined that is as “close” to A as possible. How would you do that?